ELSEVIER

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Functional Analysis 262 (2012) 1251–1273

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Invariant subspace problem for classical spaces of functions

Michał Goliński [1]

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland

Received 10 May 2011; accepted 2 November 2011

Available online 16 November 2011

Communicated by G. Schechtman

Abstract

We construct continuous linear operators without non-trivial invariant subspaces on several classical non-Banach spaces of analysis: the Schwartz space of rapidly decreasing functions, the space of smooth functions on a compact smooth manifold, the space of holomorphic functions on the unit disc or on an arbitrary polydisc, the space of entire functions on $\mathbb{C}^d$ for $d \geqslant 2$. As these are reflexive spaces, the result gives an analogous statement for the dual spaces, e.g. the space of tempered distributions. The construction works with Köthe sequence spaces and is based on methods developed by Read to construct his famous example on the space l_1.

Keywords: Invariant subspaces; Sequence spaces; Köthe spaces

1. Introduction

Existence of non-trivial invariant subspaces for operators on finite dimensional spaces is a simple consequence of the fundamental theorem of algebra, but operators on infinite dimensional spaces pose a much more difficult problem.

The question of existence of non-trivial invariant subspace for continuous operators on the Hilbert space l_2 was posed by von Neumann in the 1930s, who gave an affirmative answer for

E-mail address: golinski@amu.edu.pl.

[1] The author is supported by National Science Centre (Poland) grant No. N N201 605340.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.11.003

compact operators, this result was never published. Later many mathematicians have found invariant subspaces for some classes of Hilbert space operators – see [5,3].

First example of an operator an a Banach space without non-trivial invariant subspaces was given by Enflo in the late 1970s, construction published much later in [4]. Subsequent counterexamples in this direction were given by Read in [9]. Both examples concern artificial, *ad-hoc* constructed spaces. Later Read refined his method and proved, that an operator with a much stronger property – not having non-trivial invariant subsets – can be constructed on a large class of (non-reflexive) Banach spaces, including the space l_1, see [11]. Further development by Śliwa made it possible to find explicit sequences which give rise to operators without non-trivial invariant subspaces on the space l_1 and to construct such operators on some non-archimedean Banach spaces, see [13,14].

In the meantime, in 1983 Atzmon, using different ideas, constructed an operator without invariant subspaces on some (once again artificial) nuclear Fréchet space – see [1].

In this paper, using the ideas from the papers of Read, we construct continuous linear operators without non-trivial invariant subspaces on a range of Köthe spaces $\lambda^1(A)$, including many classical power series spaces (see Theorem 1 and Corollaries 3, 4).

One of the most interesting examples of the spaces in question is the space s of rapidly decreasing sequences which plays an important role in the structural theory of nuclear spaces.

Many classical nuclear Fréchet function spaces can be represented as Köthe sequence spaces (see [12,7]) and in most cases these representations satisfy assumptions of our Theorem 1. A list is given in Section 3. Let us emphasise, that contrary to Atzmon's construction, our construction gives rise to operators without non-trivial invariant subspaces on *natural* nuclear Fréchet spaces.

One of the intriguing exceptions is the space $H(\mathbb{C})$ of entire functions in one complex variable, for which we do not know if such an operator exists.

As these spaces are reflexive, we also get operators without non-trivial invariant subspaces on corresponding (DF)-spaces.

The proof is modelled after Read's proof as exposed in the last chapter of the book [2], including most notations and the basic lemma. Our construction is entirely self-contained, nevertheless reading the book will probably provide more insight into the workings of the, admittedly quite complicated at first sight, proof.

2. Main result

We begin with some notation. We will denote by $\mathbb{K}$ the scalar field ($\mathbb{C}$ or $\mathbb{R}$).

For convenience, we set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \dots\}$.

A norm $\|\cdot\|$ on c_{00} is called a *weighted l_1-norm* if

$$\left\|(x_j)_{j=0}^{\infty}\right\| = \sum_{j=0}^{\infty} A_j |x_j|, \quad A_j > 0.$$

Let us recall the definition of a Köthe sequence space $\lambda^1(A)$ (see [7, Ch. 27]).

An infinite matrix $A = [A_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ of non-negative numbers is called a *Köthe matrix* if the following two conditions are satisfied:

$$\forall j \in \mathbb{N}\ \exists N \in \mathbb{N}_+ \quad A_{N,j} > 0,$$
$$\forall j \in \mathbb{N}\ \forall N \in \mathbb{N}_+ \quad A_{N,j} \leqslant A_{N+1,j}.$$

Given a Köthe matrix A we define for $1 \leqslant p < \infty$

$$\lambda^p(A) = \left\{ (x_j)_{j=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}\colon\ \|x\|_N = \left(\sum_{j=0}^{\infty} \big(|x_j| A_{N,j}\big)^p \right)^{\frac{1}{p}} < \infty \text{ for all } N \in \mathbb{N}_+ \right\}$$

and call $\lambda^p(A)$ a *Köthe sequence space* associated with the matrix A. If A is a Köthe matrix, then all these spaces are Fréchet spaces with their natural locally convex topologies given by the sequence of norms $(\|\cdot\|_N)_{N\in\mathbb{N}_+}$ (see [7, 27.1]). It is clear that finite vectors form a dense subspace of $\lambda^p(A)$.

Let E be a linear topological space and let $S : E \to E$ be a continuous linear operator. A closed subspace $M \subseteq E$ will be called an *invariant subspace* of S if $S(M) \subseteq M$. Subspaces E and $\{0\}$ are called the *trivial invariant subspaces*. Any vector $x \in E$ such that $\operatorname{span}\{x, Sx, S^2x, S^3x, \ldots\}$ is dense in E is called a *cyclic vector* with respect to S. It is easily seen that an operator has no non-trivial invariant subspaces if and only if every non-zero vector is cyclic with respect to this operator.

Our main result is:

Theorem 1. *Let $A = [A_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ be a positive Köthe matrix satisfying the following conditions*:

$$\exists M\ \forall N \in \mathbb{N}_+ \quad \limsup_{j\to\infty} \frac{A_{N,j+1}}{A_{N,j}} \leqslant M, \tag{1}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j\to\infty} \frac{A_{N,j}}{A_{N+1,j}} = 0, \tag{2}$$

$$\forall N \in \mathbb{N}_+ \quad \text{the sequence } (A_{N,j})_j \text{ tends monotonically to } +\infty. \tag{3}$$

Then there exists a continuous linear operator $T : \lambda^1(A) \to \lambda^1(A)$ with no non-trivial invariant subspaces.

The main part of this paper consists of the proof of this theorem, but first we collect some corollaries.

3. Corollaries

The following result is well known.

Proposition 2. *(See [7, 27.16, 28.16].) Let A be a Köthe matrix, then the following are equivalent*:

a) *$\lambda^p(A)$ is nuclear for some $p \in [1,\infty)$*;
b) *there are $p, r \in [1,\infty)$, $p \neq r$ such that $\lambda^p(A) = \lambda^r(A)$ as vector spaces*;
c) *$\lambda^p(A) = \lambda^r(A)$ as Fréchet spaces for all $p, r \in [1,\infty)$*;
d) *for each $N \in \mathbb{N}_+$ there exists a $K \in \mathbb{N}_+$ such that*

$$\sum_{j=0}^{\infty} \frac{A_{N,j}}{A_{K,j}} < \infty.$$

3.1. Power series spaces

Let $\alpha = (\alpha_j)_{j=0}^{\infty}$ be a sequence tending monotonically to $+\infty$. For $r \in \mathbb{R} \cup \{\infty\}$ we put

$$\Lambda_r(\alpha) = \left\{ (x_j)_{j=0}^{\infty} \in \mathbb{K}^{\mathbb{N}} \colon \ \|x\|_t = \left(\sum_{j=0}^{\infty} |x_j|^2 e^{2t\alpha_j} \right)^{\frac{1}{2}} < \infty \text{ for all } t < r \right\}.$$

Observe that $\Lambda_r(\alpha) = \lambda^2([e^{t_N \alpha_j}])$, where (t_N) is a strictly increasing sequence tending to r. It can be shown (see [7, 29.3]) that:

- for any sequences α, β and any $r \in \mathbb{R}$, $\Lambda_r(\alpha) \neq \Lambda_\infty(\beta)$;
- for any sequence α and any $r_1, r_2 \in \mathbb{R}$, $\Lambda_{r_1}(\alpha) \cong \Lambda_{r_2}(\alpha)$.

Hence there are only two essentially distinct cases. The space $\Lambda_1(\alpha)$ is called a *power series space of finite type* and $\Lambda_\infty(\alpha)$ – a *power series space of infinite type*.

3.1.1. Power series spaces of finite type

Corollary 3. *Let $\alpha = (\alpha_j)$ be a sequence of non-negative numbers tending monotonically to $+\infty$ such that $\lim_j \frac{\log j}{\alpha_j} = 0$ and $\sup_j(\alpha_{j+1} - \alpha_j) < \infty$. Then there exists a continuous linear operator $T : \Lambda_1(\alpha) \to \Lambda_1(\alpha)$ with no non-trivial invariant subspaces.*

Proof. By [7, 29.6] we get that $\Lambda_1(\alpha)$ is nuclear, so by Proposition 2

$$\Lambda_1(\alpha) = \lambda^1\big(\big[e^{(1-\frac{1}{N+1})\alpha_j}\big]\big).$$

The matrix $[e^{(1-\frac{1}{N+1})\alpha_j}]$ satisfies conditions (2) and (3). One can easily check, that condition (1) is satisfied if and only if $\sup_j(\alpha_{j+1} - \alpha_j) < \infty$. The claim follows by Theorem 1. □

By Corollary 3, the following power series spaces admit an operator without non-trivial invariant subspaces:

a) Take $\alpha_j = j$. The space $\Lambda_1(j)$ is isomorphic to the space $H(\mathbb{D})$ of holomorphic functions on the unit disc (see [7, 29.4]). In fact $H(\mathbb{D}) \cong H(U)$, whenever $U \subseteq \mathbb{C}$ is open, has a finite number of connected components and is regular, i.e. Dirichlet problem is solvable in U – see [17].
b) Take $\alpha_j = \sqrt[d]{j}$, where $d \geqslant 2$. The space $\Lambda_1(\sqrt[d]{j})$ is isomorphic to the space $H(\mathbb{D}^d)$ of holomorphic functions on the d-dimensional polydisc (see [12, 8.3.2]). In fact $H(\mathbb{D}^d) \cong H(U)$, whenever $U \subseteq \mathbb{C}^d$ is open, has a finite number of connected components and is sharply pseudoconvex, i.e. there exists an open neighbourhood V of U and a continuous plurisubharmonic function ψ on V such that $U = \{z\colon \psi(z) < 0\}$ and $\{z\colon \psi(z) \leqslant 0\} \Subset V$ – see [18].

3.1.2. Power series spaces of infinite type

Corollary 4. *Let $\alpha = (\alpha_j)$ be a sequence of non-negative numbers tending monotonically to $+\infty$ such that $\sup_j \frac{\log j}{\alpha_j} < +\infty$ and $\lim_j(\alpha_{j+1} - \alpha_j) = 0$. Then there exists a continuous linear operator $T : \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$ with no non-trivial invariant subspaces.*

Proof. By [7, 29.6] we get that $\Lambda_\infty(\alpha)$ is nuclear, so by Proposition 2, $\Lambda_1(\alpha) = \lambda^1([e^{N\alpha_j}])$.

The matrix $[e^{N\alpha_j}]$ satisfies conditions (2) and (3). One can easily check, that condition (1) is satisfied if and only if $\lim_j(\alpha_{j+1} - \alpha_j) = 0$. The claim follows by Theorem 1. □

By Corollary 4 the following power series spaces admit an operator without non-trivial invariant subspaces:

a) Take $\alpha_j = \sqrt[d]{j}$, where $d \geqslant 2$. The space $\Lambda_\infty(\sqrt[d]{j})$ is isomorphic to the space $H(\mathbb{C}^d)$ of entire holomorphic functions on $\mathbb{C}^d$ – see [12, 8.3.2].
b) Take $\alpha_j = \log j$. The space $\Lambda_\infty(\log j)$ is usually denoted by s and it is called the *space of rapidly decreasing sequences*. Many spaces of analysis are isomorphic to s, including:
   - the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ – the test function space for the space of tempered distributions – see [7, 31.14];
   - the space $C^\infty[0, 1]$ – for a direct proof see [7, 29.5.4];
   - the space $\mathcal{D}_K$ of smooth functions with their support contained in a compact set $K \subset \mathbb{R}^n$, when K has a nonempty interior – see [16];
   - the space $C^\infty(K)$ for each compact C^∞-manifold K – see [16];
   - the space of all entire Dirichlet series, i.e. space of sequences (a_n) such that the series

$$\sum_{n=0}^{\infty} a_n n^z$$

   is convergent for any $z \in \mathbb{C}$ – see [12, 8.4.1].

An important omission from this list is the space $H(\mathbb{C}) \cong \Lambda_\infty(j)$. See Remark 23 below for further information.

3.2. Dual spaces

Proposition 5. *If X is a reflexive locally convex space and $T : X \to X$ is a linear and continuous operator without non-trivial invariant subspaces, then $T' : X' \to X'$ has no non-trivial invariant subspaces.*

Proof. With the usual identification in mind, we have that $T'' = T$. It is easy to check that if M is a non-trivial invariant subspace for T, then $M^\perp$ is a non-trivial invariant subspace for T'. This yields our claim. □

Corollary 6. *On the following spaces there exists a continuous linear operator without non-trivial invariant subspaces*:

a) *the space* s', *which is isomorphic, in particular, to*:
   - *the space* $\mathcal{S}'(\mathbb{R}^n)$ *of tempered distributions*;
   - *the space* $\mathcal{C}^\infty(\mathbb{T}^d)'$ *of distributions on the* d*-dimensional torus*;

b) *the space* $H(\{0\}^d)$ *of* d*-dimensional germs of holomorphic functions, where* $d \geqslant 2$ (*by* [15], $H(\mathbb{C}^d)' \cong H(\{0\}^d)$);

c) *the space* $H(\bar{\mathbb{D}})$ *of germs of holomorphic functions in the neighbourhood of the closed unit disc* (*by the well-known Köthe–Grothendieck–da Silva duality, see* [6, *Ch.* 9], $H(\mathbb{D})' \cong H(\bar{\mathbb{D}})$).

4. Notations

Throughout we will denote by c_{00} the space of all finite sequences – a linear subspace of $\mathbb{K}^{\mathbb{N}}$. Canonical basis of c_{00} will be denoted by $(e_0, e_1, e_2, \ldots)$ and $E_n = \operatorname{span}\{e_0, e_1, \ldots, e_n\}$.

For a linear basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of E_n and

$$E_n \ni x = \sum_{i=k}^{n} x_i \gamma_i, \qquad x_k \neq 0,$$

we write $\operatorname{val}_{\widetilde{\gamma}}(x) = k$. For a set $K \subseteq E_n$ we write $\operatorname{val}_{\widetilde{\gamma}}(K) = \sup_{y \in K} \operatorname{val}_{\widetilde{\gamma}}(y)$.

Remark 7. Observe that if $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ and $\widetilde{\mu} = (\gamma_0, \gamma_1, \ldots, \gamma_n, \mu_{n+1}, \ldots, \mu_m)$ are bases of E_n and E_m, respectively, then for $y \in E_n$

$$\operatorname{val}_{\widetilde{\gamma}}(y) = \operatorname{val}_{\widetilde{\mu}}(y).$$

The following simple observation will be important for us:

Proposition 8. *Let* E *be a linear topological space and let* $T : E \to E$ *be a continuous linear operator. Let* y *be a cyclic vector with respect to* T*. Then for every vector* $x \in E$*, if* $y \in \overline{\operatorname{span}\{x, Tx, T^2x, \ldots\}}$*, then* x *is also cyclic with respect to* T*.*

Proof. Take any element $z \in E$ and any neighbourhood U of z. Since y is cyclic, for some polynomial P we have that $P(T)y \in U$. However, $P(T)$ is also a continuous operator, so for some neighbourhood V of y we have that $P(T)(V) \subseteq U$. By hypothesis, we can find a polynomial Q such that $Q(T)x \in V$. Then $(P \circ Q)(T)x \in U$. $\square$

Further on, for a polynomial $P \in \mathbb{K}[t]$ with $P(t) = \sum_{l \in M} p_l t^l$, $M \subseteq \mathbb{N}$, we will write shortly that $\operatorname{sc} P \subseteq M$ (sc stands for the 'support of the sequence of coefficients') and $|P| = \sum_{l \in M} |p_l|$.

5. The matrix

First, we show that we can get much better properties of the matrix A in Theorem 1 without changing the space $\lambda^1(A)$.

Lemma 9. *If a Köthe matrix $B = [B_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ of positive numbers satisfies*

$$\exists M\ \forall N \in \mathbb{N}_+ \quad \limsup_{j\to\infty} \frac{B_{N,j+1}}{B_{N,j}} \leqslant M, \tag{4}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j\to\infty} \frac{B_{N,j}}{B_{N+1,j}} = 0, \tag{5}$$

$$\forall N \in \mathbb{N}_+ \quad \textit{sequence } (B_{N,j})_j \textit{ tends monotonically to } +\infty, \tag{6}$$

then there exist a Köthe matrix $A = [A_{N,j}]_{N\in\mathbb{N}_+, j\in\mathbb{N}}$ such that $\lambda^1(B) = \lambda^1(A)$ (topologically) and a number $Q > 1$ such that

$$\forall N \in \mathbb{N}_+\ \forall j \in \mathbb{N} \quad A_{N,j} \geqslant 1, \tag{7}$$

$$\forall N \in \mathbb{N}_+\ \forall j \in \mathbb{N} \quad \frac{A_{N,j+1}}{A_{N,j}} \leqslant Q, \tag{8}$$

$$\forall N \in \mathbb{N}_+\ \forall h \in \mathbb{N} \quad \lim_{j\to\infty} \frac{A_{N,j+h}}{A_{N+1,j}} = 0, \tag{9}$$

$$\forall N \in \mathbb{N}_+ \quad \textit{sequence } (A_{N,j})_j \textit{ tends monotonically to } +\infty, \tag{10}$$

$$\forall N \in \mathbb{N}_+ \quad \lim_{j\to\infty} \frac{Q^j}{A_{N,j}} = +\infty \tag{11}$$

and the sequence of the unit balls $(U_N)_{N\in\mathbb{N}_+}$, $U_N = \{x = (x_j) \in \mathbb{K}^{\mathbb{N}}\colon\ \|x\|_N = \sum_{j=0}^{\infty} |x_j| A_{N,j} \leqslant 1\}$, forms a basis of neighbourhoods of zero in $\lambda^1(A)$.

Proof. Take any number $Q > \max(M, 1)$, where M satisfies (4).

Let $(k_N)_{N\in\mathbb{N}_+}$ be an increasing sequence of positive integers chosen so that for $j \geqslant k_N$

$$B_{N,j} \geqslant 1 \quad \text{and} \quad \frac{B_{N,j+1}}{B_{N,j}} \leqslant Q \quad \text{and} \quad \frac{B_{N,j}}{B_{N+1,j}} < \frac{1}{2}.$$

This can be done by (6), (4) and (5), respectively. We define $A_{N,j} = B_{N,j+k_N}$. Then $A = [A_{N,j}]$ fulfils (8), (10) and, since $Q > M$, (4) implies (11).

For any $h \geqslant 0$ we have that

$$\frac{A_{N,j+h}}{A_{N+1,j}} < Q^h \frac{A_{N,j}}{A_{N+1,j}} = Q^h \frac{B_{N,j+k_N}}{B_{N+1,j+k_{N+1}}} \leqslant Q^h \frac{B_{N,j+k_N}}{B_{N+1,j+k_N}} \xrightarrow[j\to\infty]{} 0,$$

which shows (9). By the choice of k_N, we have that

$$\frac{A_{N,j}}{A_{N+1,j}} = \frac{B_{N,j+k_N}}{B_{N+1,j+k_{N+1}}} \leqslant \frac{B_{N,j+k_N}}{B_{N+1,j+k_N}} < \frac{1}{2},$$

so $A_{N,j} < \frac{1}{2^K} A_{N+K,j}$. It follows that $U_{N+K} \subseteq \frac{1}{2^K} U_N$ and by the definition of the topology on $\lambda^1(A)$, $(U_N)_{N\in\mathbb{N}_+}$ is a basis of neighbourhoods of zero.

We will show that the formal identity

$$\lambda^1(B) \ni (x_j) \longmapsto (x_j) \in \lambda^1(A)$$

is an isomorphism. We have, by the monotonicity of $(B_{N,j})_j$, that

$$\begin{aligned}\|(x_j)\|_N^{\lambda^1(A)} &= \sum_{j=0}^{\infty} |x_j| A_{N,j} = \sum_{j=0}^{\infty} |x_j| B_{N,j+k_N} \\ &\geqslant \sum_{j=0}^{\infty} |x_j| B_{N,j} = \|(x_j)\|_N^{\lambda^1(B)}.\end{aligned}$$

Similarly, by (4),

$$\begin{aligned}\|(x_j)\|_N^{\lambda^1(A)} &= \sum_{j=0}^{\infty} |x_j| A_{N,j} = \sum_{j=0}^{\infty} |x_j| B_{N,j+k_N} \\ &\leqslant \sup_j \frac{B_{N,j+k_N}}{B_{N,j}} \sum_{j=0}^{\infty} |x_j| B_{N,j} = C_N \|(x_j)\|_N^{\lambda^1(B)}. \qquad \square\end{aligned}$$

From now on, we fix a Köthe matrix A satisfying the claims of Lemma 9 and a sequence of l_1-norms with respect to this matrix, i.e. further on

$$\|(x_j)\|_N = \sum_{j=0}^{\infty} |x_j| A_{N,j}.$$

Remark 10. In the sequel we will give hints when we are using conditions in Lemma 9 with appropriate numbers, but conditions (7) and (10) will be used sometimes without notice.

6. Perturbed weighted forward shifts

Let $T : c_{00} \to c_{00}$ be a linear operator. We will call T a *perturbed weighted forward shift* if

$$T^j e_0 = \sum_{i=0}^{j} \alpha_{j,i} e_i, \quad \alpha_{j,j} \neq 0.$$

Similarly, a sequence of vectors $(\gamma_0, \gamma_1, \ldots)$ – finite or infinite – will be called a *perturbed canonical basis* if

$$\gamma_j = \sum_{i=0}^{j} \gamma_{j,i} e_i, \quad \gamma_{j,j} \neq 0.$$

Observe that $(\gamma_0, \gamma_1, \ldots)$ is a linear basis of c_{00}, moreover, for any $n \in \mathbb{N}$ the sequence $(\gamma_0, \gamma_1, \ldots, \gamma_n)$ is a linear basis of E_n.

We will work with perturbed weighted forward shifts of a particular type.

Proposition 11. *Assume that a linear map $T : c_{00} \to c_{00}$ satisfies for all $j \in \mathbb{N}$*

$$T^j e_0 = \delta_j e_j + \sigma_j T^{j-p_j} e_0, \quad \text{for } j = 1, 2, 3, \dots, \tag{12}$$

where $(\delta_j) \subset \mathbb{R} \setminus \{0\}$, $(\sigma_j) \subset \mathbb{R}$, $(p_j) \subset \mathbb{N}$ and $p_j \leqslant j$. Then:

(i) *vectors $T^j e_0$ are linearly independent*;
(ii) $\operatorname{span}\{e_0, Te_0, \dots, T^n e_0\} = \operatorname{span}\{e_0, e_1, \dots, e_n\}$;
(iii) $\operatorname{span}\{e_0, Te_0, T^2 e_0, \dots\} = c_{00}$;
(iv) $e_j = \frac{1}{\delta_j}(T^j e_0 - \sigma_j T^{j-p_j} e_0)$;
(v) $Te_j = \frac{\delta_{j+1}}{\delta_j} e_{j+1} + \frac{\sigma_{j+1}}{\delta_j} T^{j+1-p_{j+1}} e_0 - \frac{\sigma_j}{\delta_j} T^{j+1-p_j} e_0$.

Proof. The claims (i)–(iii) are obvious because the numbers δ_j in (12) are all non-zero.
The claim (iv) follows easily from (12). Applying T to (iv) yields (v). □

For perturbed weighted forward shifts we have the following crucial lemma. Its idea can be traced to [10, Lemma 5.1], but our formulation, exposition (and the proof) is an almost verbatim repetition of that in [2, Ch. 12]. We give a complete proof for the convenience of the reader.

Lemma 12. *Assume that for some integers $a, \Delta > 0$ there is given a perturbed canonical basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_{a+\Delta-1})$ of $E_{a+\Delta-1}$, with $\gamma_a = \varepsilon e_a + e_0$ and $\gamma_0 = e_0$. Let $\|\cdot\|$ be any weighted l_1-norm on c_{00} and K be any compact set with respect to the topology induced by $\|\cdot\|$ satisfying*

$$K \subseteq \{ y \in E_{a+\Delta-1} \colon \operatorname{val}_{\widetilde{\gamma}}(y) \leqslant a \}.$$

Let $v := a - \operatorname{val}_{\widetilde{\gamma}}(K)$. Then there is a constant $C = C(\widetilde{\gamma}, K, \|\cdot\|)$ such that for any $y \in K$ there is a polynomial P with $\operatorname{sc} P \subseteq [v, a + \Delta)$ and $|P| \leqslant C$ such that for each perturbed weighted forward shift $T : c_{00} \to c_{00}$ with

$$T^j e_0 = \gamma_j, \quad \text{if } j = 1, 2, \dots, a + \Delta - 1 \tag{13}$$

and

$$T^{b+v+j} e_0 = e_{b+v+j} + DT^{v+j} e_0, \quad \text{if } j = 0, 1, \dots, a + \Delta - v - 1 \tag{14}$$

for some integer $b > a + \Delta$ and some number $D > 0$, we have that

$$\left\| \frac{T^b}{D} P(T) y - e_0 \right\| \leqslant 2\varepsilon \|e_a\| + \frac{C}{D} \Big(\max_{b+v \leqslant i < b+a+\Delta} \|e_i\| + \max_{b+a+\Delta \leqslant j \leqslant b+2(a+\Delta-1)} \|T^j e_0\| \Big).$$

Proof. Define a linear operator $T' : E_{a+\Delta-1} \to E_{a+\Delta-1}$ by the relation

$$T'(\gamma_j) = \begin{cases} \gamma_{j+1}, & \text{if } j < a + \Delta - 1, \\ 0, & \text{if } j = a + \Delta - 1. \end{cases}$$

Take any $z \in E_{a+\Delta-1}$, $z \neq 0$. Then the vectors

$$z, T'z, T'^2 z, \ldots, T'^{a+\Delta-1-\mathrm{val}_{\widetilde{\gamma}}(z)} z$$

form a linear basis of $\mathrm{span}\{\gamma_{\mathrm{val}_{\widetilde{\gamma}}(z)}, \gamma_{\mathrm{val}_{\widetilde{\gamma}}(z)+1}, \ldots, \gamma_{a+\Delta-1}\}$. In particular, if $z \in K$, then $\mathrm{val}_{\widetilde{\gamma}}(z) \leqslant a$ and, consequently, there is a polynomial P_z with $\mathrm{sc}\, P_z \subseteq [a - \mathrm{val}_{\widetilde{\gamma}}(z), a + \Delta)$ such that

$$P_z(T')z = \gamma_a = \varepsilon e_a + e_0.$$

For some neighbourhood B_z of z in $E_{a+\Delta-1}$ we have therefore

$$\| P_z(T')y - e_0 \| < 2\varepsilon \|e_a\|, \quad \text{for every } y \in B_z.$$

By the compactness of K we get that for any $y \in K$ there is a polynomial P for which

$$\| P(T')y - e_0 \| < 2\varepsilon \|e_a\| \tag{15}$$

and $\mathrm{sc}\, P \subseteq [v, a + \Delta)$, $|P| \leqslant C$ for some $C > 0$.

Now let $T : c_{00} \to c_{00}$ be any linear operator satisfying (13) and (14).

Fix $y = \sum_{k=0}^{a+\Delta-1} y_k \gamma_k \in K$ and fix $P(t) = \sum_{l=v}^{a+\Delta-1} p_l t^l$ chosen so that (15) holds. Then

$$P(T)y = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} y_k p_l T^l \gamma_k = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} y_k p_l T^{k+l} e_0 = \sum_{j=v}^{2(a+\Delta-1)} \lambda_j T^j e_0, \tag{16}$$

where

$$\lambda_j = \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{j,k+l} y_k p_l, \tag{17}$$

where $\delta_{i,j}$ is the Kronecker delta. By the boundedness of K, we can assume (taking larger C if necessary) that $\sum_{j=v}^{2(a+\Delta-1)} |\lambda_j| \leqslant C$.

We get that

$$P(T')y = P(T') \sum_{k=0}^{a+\Delta-1} y_k \gamma_k = \sum_{j=v}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \sum_{l=v}^{a+\Delta-1} \delta_{k+l,j} y_k p_l \gamma_j \overset{(17)}{=} \sum_{j=v}^{a+\Delta-1} \lambda_j \gamma_j$$

$$\overset{(13)}{=} \sum_{j=v}^{a+\Delta-1} \lambda_j T^j e_0 = \sum_{j=0}^{a+\Delta-v-1} \lambda_{j+v} T^{j+v} e_0. \tag{18}$$

It follows, by (13), that

$$\begin{aligned}\frac{T^b}{D}P(T')y &= \sum_{j=0}^{a+\Delta-v-1}\lambda_{j+v}\frac{T^{b+v+j}e_0}{D}\\ &= \sum_{j=0}^{a+\Delta-v-1}\frac{\lambda_{j+v}}{D}e_{b+v+j} + \sum_{j=0}^{a+\Delta-v-1}\lambda_{j+v}T^{j+v}e_0\\ &= \sum_{j=0}^{a+\Delta-v-1}\frac{\lambda_{j+v}}{D}e_{b+v+j} + P(T')y.\end{aligned}$$

Therefore

$$\left\|\frac{T^b}{D}P(T')y - P(T')y\right\| \leqslant \frac{C}{D}\cdot \sup_{b+v\leqslant j<b+a+\Delta}\|e_j\|. \tag{19}$$

However, by (16) and (18), we get that

$$T^bP(T)y - T^bP(T')y = T^b\sum_{j=a+\Delta}^{2(a+\Delta-1)}\lambda_jT^je_0,$$

so

$$\left\|\frac{T^b}{D}P(T)y - \frac{T^b}{D}P(T')y\right\| \leqslant \frac{C}{D}\cdot \sup_{b+a+\Delta\leqslant j\leqslant b+2(a+\Delta-1)}\|T^je_0\|. \tag{20}$$

Taking (15), (19) and (20) into account we get the final conclusion. □

7. The operator

In this section we define an operator, which under some conditions, does not have non-trivial invariant subspaces. Motivated by Lemma 12, assume that there are given sequences (Δ_n), (a_n), (b_n) such that

$$\begin{aligned}1 &= \Delta_1 < a_1 < a_1+\Delta_1 < b_1 < b_1+a_1+\Delta_1\\ &= \Delta_2 < a_2 < a_2+\Delta_2 < b_2 < b_2+a_2+\Delta_2\\ &= \Delta_3 < \cdots,\end{aligned}$$

a sequence $(v_n)\subseteq\mathbb{N}$ satisfying $v_n < a_n+\Delta_n$ and numbers $\alpha_j>0$ for $j\in[\Delta_n,a_n)\cup[a_n+\Delta_n, b_n+v_n)$. Then we define a perturbed weighted forward shift $T: c_{00}\to c_{00}$ by the relation

$$T^je_0 = \begin{cases}\frac{1}{A_{N_n,a_n}}e_j + T^{j-a_n}e_0, & \text{if } j\in[a_n,a_n+\Delta_n),\\ e_j + Q^{b_n}T^{j-b_n}e_0, & \text{if } j\in[b_n+v_n, b_n+a_n+\Delta_n),\\ \alpha_je_j, & \text{otherwise},\end{cases} \tag{21}$$

where

$$(N_n) = (1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots)$$

is a sequence containing any positive integer infinitely many times and Q is the number from Lemma 9.

The perturbed weighted forward shift defined by (21) satisfies the following corollary.

Corollary 13. *By Proposition* 11(v), *we get that* (21) *implies the following*:

$$Te_j = \begin{cases} \alpha_1 e_1, & \text{if } j = 0, \\ \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}, & \text{if } j \in [\Delta_n, a_n - 1), \\ \frac{1}{\alpha_{a_n - 1}} (\frac{1}{A_{N_n, a_n}} e_{a_n} + e_0), & \text{if } j = a_n - 1, \\ e_{j+1}, & \text{if } j \in [a_n, a_n + \Delta_n - 1), \\ \alpha_{a_n+\Delta_n} A_{N_n, a_n} e_{a_n+\Delta_n} - A_{N_n,a_n} T^{\Delta_n} e_0, & \text{if } j = a_n + \Delta_n - 1, \\ \frac{\alpha_{j+1}}{\alpha_j} e_{j+1}, & \text{if } j \in [a_n + \Delta_n, b_n + v_n - 1), \\ \frac{1}{\alpha_{b_n+v_n-1}} (e_{b_n+v_n} + Q^{b_n} T^{v_n} e_0), & \text{if } j = b_n + v_n - 1, \\ e_{j+1}, & \text{if } j \in [b_n + v_n, b_n + a_n + \Delta_n - 1), \\ \alpha_{b_n+a_n+\Delta_n} e_{b_n+a_n+\Delta_n} - Q^{b_n} \alpha_{a_n+\Delta_n} e_{a_n+\Delta_n}, & \text{if } j = b_n + a_n + \Delta_n - 1. \end{cases}$$

By a suitable choice of parameters, we will ensure that T is a continuous linear operator on $\lambda^1(A)$ without non-trivial invariant subspaces. The seemingly more complicated Fréchet space structure allows us to avoid the complicated “third step” in the Bayart–Matheron approach and yields a simpler proof. For this reason this proof *would not work* in a Banach space setting.

As the proof is rather lengthy we give an outline for the convenience of the reader. Let us denote

$$l_{1,N} := \left\{ x = (x_j)_{j \in \mathbb{N}} \colon \sum_{j=0}^{\infty} |x_j| A_{N,j} < +\infty \right\}.$$

What we prove, in fact, is, that with a suitable choice of all the parameters:

- the operator T is bounded on each of the spaces $l_{1,N}$ and can therefore be uniquely extended to, say, $T_N : l_{1,N} \to l_{1,N}$;
- the vector e_0 is cyclic with respect to T_N;
- if $x \in l_{1,N+1}$ and $\|x\|_1 = 1$, then there is a polynomial P such that

$$\left\| P(T)x - e_0 \right\|_N \leqslant 6.$$

In the light of Proposition 8, the last condition states that while x is not necessarily cyclic for T_N, it has some property in this direction. Observe that because $\lambda^1(A) = \bigcap_{N \in \mathbb{N}_+} l_{1,N}$ and the norms $(\| \cdot \|_N)_{N \in \mathbb{N}_+}$ form a fundamental system, this is indeed enough to prove our goal.

8. Choosing the parameters

Recall that we have already constructed in Lemma 9 a Köthe matrix A satisfying (8)–(11) with some number $Q > 1$. We will now inductively choose all the necessary parameters for the definition (21).

The guiding principle is, that once the sequence $(a_1, b_1, a_2, b_2, \ldots)$ increases sufficiently rapidly, the operator T has no non-trivial invariant subspaces. We discuss what we mean by "sufficiently rapidly" in this section.

Our "unit of work" is the interval $[\Delta_n, b_n + a_n + \Delta_n)$. As the construction for consecutive intervals ($n \geqslant 2$) is very similar to the first interval $[\Delta_1, b_1 + a_1 + \Delta_1)$ we present it in a unified way, indicating the differences where necessary.

Assume now that we have already found all the parameters $\Delta_{n-1}, a_{n-1}, b_{n-1}, v_{n-1}$ with

$$\Delta_{n-1} < a_{n-1} < a_{n-1} + \Delta_{n-1} < b_{n-1} < b_{n-1} + v_{n-1} < b_{n-1} + a_{n-1} + \Delta_{n-1}$$

and the numbers α_j for $j \in [\Delta_{n-1}, a_{n-1}) \cup [a_{n-1} + \Delta_{n-1}, b_{n-1} + v_{n-1})$. We will show how to find suitable numbers Δ_n, a_n, b_n, v_n and α_j for $j \in [\Delta_n, a_n) \cup [a_n + \Delta_n, b_n + v_n)$.

First, put $\Delta_n = b_{n-1} + a_{n-1} + \Delta_{n-1}$ (in the case of the first interval we put $\Delta_1 = 1$).

If $n \geqslant 2$, let us choose, by (9), a natural number c_{n-1} such that

$$\frac{A_{N_{n-1},j}}{A_{N_{n-1}+1,j}} \leqslant \frac{1}{(2Q^4)^{b_{n-1}+v_{n-1}}} \quad \text{for } j \geqslant c_{n-1} \tag{22}$$

and $c_{n-1} = \Delta_n + r_{n-1}(b_{n-1} + v_{n-1})$, for some $r_{n-1} \in \mathbb{N}$. We put $c_0 = r_0 = 1$ and $b_0 = v_0 = 0$.

We also calculate for $n \geqslant 2$:

$$\eta_{n-1} = \max\left(\sup_{\Delta_n \leqslant j < c_{n-1}} \frac{A_{N_{n-1},j+b_{n-1}+v_{n-1}}}{A_{N_{n-1}+1,j}}, 1\right), \tag{23}$$

$$\begin{aligned} s_{n-1} = \max\Bigg(& \sup_{a_{n-1}+\Delta_{n-1} \leqslant j < b_{n-1}+v_{n-1}} \frac{A_{N_{n-1},j+b_{n-1}+v_{n-1}}}{\alpha_j A_{N_{n-1}+1,j}}, \\ & \sup_{b_{n-1}+v_{n-1} \leqslant j < b_{n-1}+a_{n-1}+\Delta_{n-1}} \frac{Q^{b_{n-1}} A_{N_{n-1},j+v_{n-1}}}{A_{N_{n-1}+1,j}}\Bigg). \end{aligned} \tag{24}$$

We put $\eta_0 = s_0 = 1$.

By (11) and (9), we can choose a natural number a_n such that:

$$a_n \geqslant 4\Delta_n \quad \text{and} \quad 4 \text{ divides } a_n, \tag{25}$$

$$a_n \geqslant c_{n-1} + b_{n-1} + v_{n-1}, \tag{26}$$

$$\frac{A_{N_n,a_n}}{A_{N_n+1,a_n}} \leqslant \frac{1}{(2Q^4)^{\Delta_n} \|e_0\|_1}, \tag{27}$$

$$\frac{(Q^3)^{a_n-1-\Delta_n-(r_{n-1}+1)(b_{n-1}+v_{n-1})}}{\eta_{n-1}^{r_{n-1}} A_{N_n,a_n}} \geqslant 1, \tag{28}$$

$$A_{N_n,a_n} \geqslant s_{n-1}, \tag{29}$$

where conditions (26) and (29) can be omitted when $n = 1$.

We put

$$\alpha_{\Delta_n} = \frac{1}{A_{N_n,a_n}} \tag{30}$$

and

$$\alpha_j = \begin{cases} \alpha_{\Delta_n}, & \text{if } j \in [\Delta_n, \Delta_n + b_{n-1} + v_{n-1}); \\ \frac{1}{\eta_{n-1}}\alpha_{\Delta_n}, & \text{if } j \in [\Delta_n + b_{n-1} + v_{n-1}, \\ & \qquad \Delta_n + 2(b_{n-1} + v_{n-1})); \\ \frac{1}{\eta_{n-1}^2}\alpha_{\Delta_n}, & \text{if } j \in [\Delta_n + 2(b_{n-1} + v_{n-1}), \\ & \qquad \Delta_n + 3(b_{n-1} + v_{n-1})); \\ \vdots & \\ \frac{1}{\eta_{n-1}^{r_{n-1}}}\alpha_{\Delta_n}, & \text{if } j \in [\Delta_n + r_{n-1}(b_{n-1} + v_{n-1}), \\ & \qquad \Delta_n + (r_{n-1}+1)(b_{n-1} + v_{n-1})); \\ \frac{(Q^3)^{j-\Delta_n-(r_{n-1}+1)(b_{n-1}+v_{n-1})}}{\eta_{n-1}^{r_{n-1}}}\alpha_{\Delta_n}, & \text{if } j \in [\Delta_n + (r_{n-1}+1)(b_{n-1} + v_{n-1}), a_n). \end{cases} \tag{31}$$

Even though the rest of the parameters is not yet fixed, we can define a basis

$$\widetilde{\gamma}_n = \left(e_0, Te_0, \ldots, T^{a_n+\Delta_n-1}e_0\right)$$

of $E_{a_n+\Delta_n-1}$ using the already defined parameters and the definition (21). As $\widetilde{\gamma}_n$ is a basis, the linear projection $\tau_n : E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$

$$\sum_{i=0}^{a_n+\Delta_n-1} x_i T^i e_0 \xmapsto{\tau_n} \sum_{i=0}^{a_n/4} x_i T^i e_0 \tag{32}$$

is well defined. We define a compact set

$$K_n = \left\{ x \in E_{a_n+\Delta_n-1}\colon\ \|x\|_1 \leqslant 1 \text{ and } \|\tau_n x\|_1 \geqslant \frac{1}{2} \right\}. \tag{33}$$

By the definition (32), we have for $y \in K_n$ that $\operatorname{val}_{\widetilde{\gamma}_n}(y) \leqslant a_n/4$. In fact, it is easy to see, that $\operatorname{val}_{\widetilde{\gamma}_n}(K_n) = a_n/4$. Let us define (in accordance with the symbols used in Lemma 12)

$$v_n = a_n - \operatorname{val}_{\widetilde{\gamma}_n}(K_n) = 3a_n/4.$$

Now $a = a_n$, $\Delta = \Delta_n$, $K = K_n$, $v = v_n$ and the norm $\|\cdot\| = \|\cdot\|_{N_n}$ fulfil the assumptions of Lemma 12. Let $C_n = C(\widetilde{\gamma}_n, K_n, \|\cdot\|_{N_n})$ be the constant from Lemma 12.

Now, we fix b_n to be a natural number such that:

$$b_n \geqslant a_n + \Delta_n,$$
$$\frac{(Q^3)^{b_n+v_n-1-a_n-\Delta_n}}{Q^{2b_n}} \geqslant \max(\alpha_{v_n}, 1), \tag{34}$$
$$Q^{b_n} \geqslant \left(2Q^4\right)^{a_n+\Delta_n} A_{N_n, b_n+2(a_n+\Delta_n)} \max(C_n, 1). \tag{35}$$

This is possible as $A_{N_n, j+2(a_n+\Delta_n)}$ increases slower than Q^j by (11) and (8).

Finally, we define

$$\alpha_j = \frac{(Q^3)^{j-a_n-\Delta_n}}{Q^{b_n}}, \quad \text{for } j \in [a_n+\Delta_n, b_n+v_n). \tag{36}$$

Let us gather properties of the numbers α_j at the ends of the considered intervals.

Corollary 14. *If all the parameters are chosen as described in this section, then we have that*

$$\alpha_{\Delta_n} \overset{(30)}{=} \frac{1}{A_{N_n,a_n}} \leqslant 1; \tag{37}$$
$$\alpha_{a_n-1} \overset{(31)}{=} \frac{(Q^3)^{a_n-1-\Delta_n-(r_{n-1}+1)(b_{n-1}+v_{n-1})}}{\eta_{n-1}^{r_{n-1}} A_{N_n,a_n}} \overset{(28)}{\geqslant} 1; \tag{38}$$
$$\alpha_{a_n+\Delta_n} \overset{(36)}{=} \frac{1}{Q^{b_n}}; \tag{39}$$
$$\alpha_{b_1+v_1-1} \overset{(36)}{=} \frac{(Q^3)^{b_n+v_n-1-a_n-\Delta_n}}{Q^{b_n}} \overset{(34)}{\geqslant} Q^{b_n} \max(\alpha_{v_n}, 1) \geqslant 1; \tag{40}$$
$$\frac{\alpha_{j+1}}{\alpha_j} \overset{(31),(36)}{\leqslant} Q^3 \quad \textit{for } j \in [\Delta_n, a_n-1) \cup [a_n+\Delta_n, b_n+v_n); \tag{41}$$
$$\frac{\alpha_{j+b_n+v_n}}{\alpha_j} \overset{(31)}{\leqslant} \frac{1}{\eta_n} \quad \textit{for } j \in [\Delta_{n+1}, c_n). \tag{42}$$

Now we are in position to apply Lemma 12 to the operator T.

Corollary 15. *If all the parameters are chosen as described in this section, then for every $y \in K_n$ there is a polynomial P with* $\operatorname{sc} P \subseteq [v_n, a_n+\Delta_n)$ *and $|P| \leqslant C_n$ such that*

$$\left\| \frac{T^{b_n}}{Q^{b_n}} P(T)y - e_0 \right\|_{N_n} \leqslant 4,$$

where $n \in \mathbb{N}_+$ and $T : c_{00} \to c_{00}$ is defined in (21).

Proof. By Lemma 12 applied with $a = a_n$, $\Delta = \Delta_n$, $\widetilde{\gamma} = \widetilde{\gamma}_n$, $\varepsilon = \frac{1}{A_{N_n,a_n}}$, $K = K_n$, $\|\cdot\| = \|\cdot\|_{N_n}$, $v = v_n$, $b = b_n$, $D = Q^{b_n}$ and T as in (21), we get that for every $y \in K_n$ there is a polynomial P with $\operatorname{sc} P \subseteq [v_n, a_n+\Delta_n)$ and $|P| \leqslant C_n$ such that

$$\begin{aligned}
&\left\| \frac{T^{b_n}}{Q^{b_n}} P(T)y - e_0 \right\|_{N_n} \\
&\leqslant 2\frac{\|e_{a_n}\|_{N_n}}{A_{N_n,a_n}} + \frac{C_n}{Q^{b_n}}\Bigl(\max_{b_n+v_n\leqslant i<b_n+a_n+\Delta_n} \|e_i\|_{N_n} + \max_{b_n+a_n+\Delta_n\leqslant j\leqslant b_n+2(a_n+\Delta_n-1)} \bigl\|T^j e_0\bigr\|_{N_n}\Bigr) \\
&= 2 + \frac{C_n}{Q^{b_n}}\Bigl(A_{N_n,b_n+a_n+\Delta_n-1} + \max_{b_n+a_n+\Delta_n\leqslant j\leqslant b_n+2(a_n+\Delta_n-1)} \bigl\|T^j e_0\bigr\|_{N_n}\Bigr) \\
&\overset{(35),(21)}{\leqslant} 3 + \frac{C_n}{Q^{b_n}}\Bigl(\max_{b_n+a_n+\Delta_n\leqslant j\leqslant b_n+2(a_n+\Delta_n-1)} \|\alpha_j e_j\|_{N_n}\Bigr) \\
&\leqslant 3 + \frac{C_n A_{N_n,b_n+2(a_n+\Delta_n)}}{Q^{b_n}}\Bigl(\max_{b_n+a_n+\Delta_n\leqslant j\leqslant b_n+2(a_n+\Delta_n-1)} \alpha_j\Bigr) \\
&\overset{(35)}{\leqslant} 3 + \max_{\Delta_{n+1}\leqslant j<\Delta_{n+1}+b_n+v_n} \alpha_j \overset{(31)}{\leqslant} 3 + \alpha_{\Delta_{n+1}} \overset{(30)}{\leqslant} 4. \qquad \square
\end{aligned}$$

9. Continuity

Lemma 16. *When all the parameters are chosen in the way specified in Section* 8*, the operator T given by* (21) *is continuous with respect to the topology induced on c_{00} by $\lambda^1(A)$. In fact, for each $N \in \mathbb{N}_+$ the operator T is bounded with respect to the weighted l_1-norm $\|\cdot\|_N$:*

$$\|Tx\|_N \leqslant 2Q^4\|x\|_N.$$

Proof. Because of the special form of the weighted l_1-norms it suffices to show that for any $j \in \mathbb{N}$ and $N \in \mathbb{N}_+$

$$\frac{\|Te_j\|_N}{\|e_j\|_N} \leqslant 2Q^4.$$

To do that we calculate Te_j for various j using Corollary 13. We get that:

- If $j = 0$, then

$$\frac{\|Te_0\|_N}{\|e_0\|_N} = \frac{\|\alpha_1 e_1\|_N}{\|e_0\|_N} = \alpha_{\Delta_1}\frac{A_{N,1}}{A_{N,0}} \overset{(37),(8)}{\leqslant} Q < 2Q^4.$$

- If $j \in [\Delta_n, a_n - 1)$ or $j \in [a_n + \Delta_n, b_n + v_n - 1)$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\|\frac{\alpha_{j+1}}{\alpha_j} e_{j+1}\|_N}{\|e_j\|_N} = \frac{\alpha_{j+1}}{\alpha_j}\frac{A_{N,j+1}}{A_{N,j}} \overset{(41),(8)}{\leqslant} Q^3 \cdot Q < 2Q^4.$$

- If $j \in [a_n, a_n + \Delta_n - 1)$ or $j \in [b_n + v_n, b_n + a_n + \Delta_n - 1)$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{\|e_{j+1}\|_N}{\|e_j\|_N} = \frac{A_{N,j+1}}{A_{N,j}} \overset{(8)}{<} Q < 2Q^4.$$

- If $j = a_n - 1$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} = \frac{\|\frac{1}{\alpha_{a_n-1}}(\frac{1}{A_{N_n,a_n}}e_{a_n} + e_0)\|_N}{\|e_{a_n-1}\|_N} \leqslant \frac{A_{N,a_n}}{\alpha_{a_n-1}A_{N_n,a_n}A_{N,a_n-1}} + \frac{A_{N,0}}{\alpha_{a_n-1}A_{N,a_n-1}}.$$

By our choice of the parameters, it follows that

$$\frac{1}{\alpha_{a_n-1}A_{N_n,a_n}}\frac{A_{N,a_n}}{A_{N,a_n-1}} \overset{(38),(8)}{\leqslant} Q,$$
$$\frac{1}{\alpha_{a_n-1}}\frac{A_{N,0}}{A_{N,a_n-1}} \overset{(38),(10)}{\leqslant} 1,$$

so

$$\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} \leqslant Q + 1 < 2Q^4.$$

- If $j = a_n + \Delta_n - 1$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} = \frac{\|\alpha_{a_n+\Delta_n}A_{N_n,a_n}e_{a_n+\Delta_n} - A_{N_n,a_n}T^{\Delta_n}e_0\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \overset{(21)}{\leqslant} \frac{\alpha_{a_n+\Delta_n}A_{N_n,a_n}A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} + \frac{A_{N_n,a_n}\alpha_{\Delta_n}A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}}.$$

By our choice of the parameters, it follows that

$$\alpha_{a_n+\Delta_n}A_{N_n,a_n}\frac{A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} \overset{(39)}{=} \frac{A_{N_n,a_n}}{Q^{b_n}}\frac{A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} \overset{(35),(8)}{\leqslant} Q,$$
$$\alpha_{\Delta_n}A_{N_n,a_n}\frac{A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}} \overset{(37),(10)}{\leqslant} 1,$$

so

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \leqslant Q + 1 < 2Q^4.$$

- If $j = b_n + v_n - 1$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_{b_n+v_n-1}\|_N}{\|e_{b_n+v_n-1}\|_N} = \frac{\|\frac{1}{\alpha_{b_n+v_n-1}}(e_{b_n+v_n} + Q^{b_n}T^{v_n}e_0)\|_N}{\|e_{b_n+v_n-1}\|_N} \leqslant \frac{A_{N,b_n+v_n}}{\alpha_{b_n+v_n-1}A_{N,b_n+v_n-1}} + \frac{Q^{b_n}\alpha_{v_n}A_{N,v_n}}{\alpha_{a_{b_n+v_n-1}}A_{N,b_n+v_n-1}},$$

where $T^{v_n} = \alpha_{v_n} e_{v_n}$ because $v_n = \frac{3a_n}{4} \in [\Delta_n, a_n)$. By our choice of the parameters, it follows that

$$\frac{1}{\alpha_{b_n+v_n-1}} \frac{A_{N,b_n+v_n}}{A_{N,b_n+v_n-1}} \overset{(40),(8)}{\leqslant} Q,$$
$$\frac{Q^{b_n}\alpha_{v_n}}{\alpha_{a_{b_n+v_n-1}}} \frac{A_{N,v_n}}{A_{N,b_n+v_n-1}} \overset{(40),(10)}{\leqslant} 1,$$

so

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \leqslant Q+1 < 2Q^4.$$

- If $j = b_n + a_n + \Delta_n - 1$ for some $n \in \mathbb{N}_+$, then

$$\frac{\|Te_{b_n+a_n+\Delta_n-1}\|_N}{\|e_{b_n+a_n+\Delta_n-1}\|_N} = \frac{\|\alpha_{b_n+a_n+\Delta_n} e_{b_n+a_n+\Delta_n} - Q^{b_n}\alpha_{a_n+\Delta_n} e_{a_n+\Delta_n}\|_N}{\|e_{b_n+a_n+\Delta_n-1}\|_N}$$
$$\leqslant \frac{\alpha_{b_n+a_n+\Delta_n} A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} + \frac{Q^{b_n}\alpha_{a_n+\Delta_n} A_{N,a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}}.$$

By our choice of the parameters, it follows that

$$\alpha_{b_n+a_n+\Delta_n} \frac{A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} = \alpha_{\Delta_{n+1}} \frac{A_{N,b_n+a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} \overset{(37),(8)}{\leqslant} Q,$$
$$Q^{b_n}\alpha_{a_n+\Delta_n} \frac{A_{N,a_n+\Delta_n}}{A_{N,b_n+a_n+\Delta_n-1}} \overset{(39),(10)}{\leqslant} 1,$$

so

$$\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \leqslant Q+1 < 2Q^4. \qquad \square$$

Corollary 17. *The operator T given by* (21) *can be uniquely extended to a continuous linear operator $T : \lambda^1(A) \to \lambda^1(A)$.*

10. Cyclic vectors

We start this section with some technical preparatory facts and afterwards we show that all the non-zero vectors in $\lambda^1(A)$ are cyclic with respect to the operator T defined in (21) and extended to $\lambda^1(A)$.

In Section 8 we have defined linear projections $\tau_n : E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ – see (32). Given that $(e_0, Te_0, \ldots, T^{a_n+\Delta_n-1}e_0)$ is a perturbed weighted canonical basis, and, by (25), $\Delta_n \leqslant a_n/4$, a straightforward computation using Proposition 11(iv) yields that in the canonical basis τ_n is given by the following formula:

$$\tau_n e_j = \begin{cases} e_j, & \text{if } j \leqslant \frac{a_n}{4}, \\ 0, & \text{if } j \in (\frac{a_n}{4}, a_n), \\ -A_{N_n,a_n} T^{j-a_n} e_0, & \text{if } j \in [a_n, a_n + \Delta_n). \end{cases} \tag{43}$$

In terms of these projections we have also defined compact sets K_n – see (33).

Proposition 18. *For any $n \in \mathbb{N}_+$ the projection $\tau_n : E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ satisfies*

$$\|\tau_n x\|_1 \leqslant \|x\|_{N_n+1}.$$

Proof. Once again, by the special form of weighted l_1-norms we need to show the claim for $x = e_j$, where $j = 0, 1, \dots, a_n + \Delta_n - 1$. For $j \in [0, a_n)$ we get, by (43), that

$$\|\tau_n e_j\|_1 \leqslant \|e_j\|_1 \leqslant \|e_j\|_{N_n+1}.$$

For $j \in [a_n, a_n + \Delta_n)$ we get by Lemma 16 that

$$\frac{\|\tau_n e_j\|_1}{\|e_j\|_{N_n+1}} \overset{(43)}{=} \frac{A_{N_n,a_n}\|T^{j-a_n} e_0\|_1}{\|e_j\|_{N_n+1}} \leqslant \frac{A_{N_n,a_n}}{A_{N_n+1,a_n}} \left(2Q^4\right)^{\Delta_n} \|e_0\|_1 \overset{(27)}{\leqslant} 1. \qquad \square$$

For a set $M \subseteq \mathbb{N}$ we denote by π_M the canonical projection from $\lambda^1(A)$ onto $\operatorname{span}\{e_j\colon\ j \in M\}$. We write $\pi_m = \pi_{[0,m]}$.

Lemma 19. *Let $N \in \mathbb{N}_+$ and let (n_k) be a sequence of natural numbers such that $N_{n_k} = N$. Let $y \in \lambda^1(A)$ be a vector such that $\|y\|_1 = 1$. If parameters in the definition* (21) *are chosen as in Section* 8*, then for all but finitely many k*

$$\pi_{a_{n_k}+\Delta_{n_k}-1}(y) \in K_{n_k},$$

where K_{n_k} is defined in (33)*.*

Proof. We have that $\|\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 \leqslant \|y\|_1 \leqslant 1$, so we only need to show that

$$\left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\right\|_1 \geqslant \frac{1}{2}$$

for almost all k. By (43) and Proposition 18, we get that

$$\begin{aligned} \left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\right\|_1 &= \left\|\pi_{a_{n_k}/4}(y) + \tau_{n_k}\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\right\|_1 \\ &\geqslant \left\|\pi_{a_{n_k}/4}(y)\right\|_1 - \left\|\tau_{n_k}\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\right\|_1 \\ &\geqslant \left\|\pi_{a_{n_k}/4}(y)\right\|_1 - \left\|\pi_{[a_{n_k},a_{n_k+\Delta_{n_k}})}(y)\right\|_{N+1} \xrightarrow[k\to\infty]{} 1, \end{aligned}$$

so indeed, for k big enough, $\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}(y)\|_1 \geqslant \frac{1}{2}$. $\square$

Proposition 20. *For each $n \in \mathbb{N}_+$ we have for $j \geqslant a_n + \Delta_n$ that*

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leqslant 2.$$

Proof. The proof consists of checking all the possible cases for j.

- If $j \in [a_n + \Delta_n, b_n + v_n)$, then, by Proposition 11(iv) and (21), we get

$$T^{b_n+v_n} e_j = T^{b_n+v_n}\left(\frac{1}{\alpha_j} T^j e_0\right) = \frac{1}{\alpha_j}\alpha_{b_n+v_n+j} e_{b_n+v_n+j},$$

where the last equality is true as $b_n + a_n + \Delta_n < b_n + v_n + j < \Delta_{n+1} + b_n + v_n$. Therefore

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|Te_j\|_{N_n+1}} = \alpha_{b_n+v_n+j}\frac{A_{N_n,b_n+v_n+j}}{\alpha_j A_{N_n+1,j}} \overset{(24)}{\leqslant} \alpha_{b_n+v_n+j} s_n \overset{(31)}{=} \alpha_{\Delta_{n+1}} s_n \overset{(37),(29)}{\leqslant} 1.$$

- If $j \in [b_n + v_n, b_n + a_n + \Delta_n) = [b_n + v_n, \Delta_{n+1})$, then, by Proposition 11(iv) and (21),

$$T^{b_n+v_n} e_j = T^{b_n+v_n}\big(T^j e_0 - Q^{b_n} T^{j-b_n} e_0\big) = T^{j+b_n+v_n} e_0 - Q^{b_n} T^{j+v_n} e_0.$$

Observe that $2v_n = \frac{3}{2}a_n \overset{(25)}{\geqslant} a_n + \Delta_n$. It follows that

$$\Delta_{n+1} = b_n + a_n + \Delta_n \leqslant b_n + 2v_n \leqslant j + v_n < j + b_n + v_n < \Delta_{n+1} + b_n + v_n \tag{44}$$

and, by (21), we get

$$\begin{aligned}\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} &\overset{(44),(21)}{\leqslant} \frac{\alpha_{j+b_n+v_n} A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + \frac{\alpha_{j+v_n} Q^{b_n} A_{N_n,j+v_n}}{A_{N_n+1,j}} \\ &\overset{(44),(31)}{=} \frac{\alpha_{\Delta_{n+1}} A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} + \frac{\alpha_{\Delta_{n+1}} Q^{b_n} A_{N_n,j+v_n}}{A_{N_n+1,j}}.\end{aligned}$$

However,

$$\begin{aligned}\alpha_{\Delta_{n+1}}\frac{A_{N_n,j+b_n+v_n}}{A_{N_n+1,j}} &\overset{(8)}{\leqslant} \alpha_{\Delta_{n+1}}\frac{Q^{b_n} A_{N_n,j+v_n}}{A_{N_n+1,j}} \overset{(24)}{\leqslant} \alpha_{\Delta_{n+1}} s_n \overset{(37),(29)}{\leqslant} 1, \\ \alpha_{\Delta_{n+1}}\frac{Q^{b_n} A_{N_n,j+v_n}}{A_{N_n+1,j}} &\overset{(24)}{\leqslant} \alpha_{\Delta_{n+1}} s_n \overset{(37),(29)}{\leqslant} 1,\end{aligned}$$

so we get that

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leqslant 2.$$

- If $j \in [\Delta_{n+1}, c_n) \subseteq [\Delta_{n+1}, a_n)$, then, by Proposition 11(iv) and (21), we get that $e_j = \frac{1}{\alpha_j} T^j e_0$ and

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} = \frac{\alpha_{j+b_n+v_n}}{\alpha_j} \frac{A_{N_n, j+b_n+v_n}}{A_{N_n+1, j}} \overset{(42),(23)}{\leqslant} \frac{1}{\eta_n} \eta_n = 1,$$

because, by (26), $j + b_n + v_n < a_{n+1}$.

- If $j \geqslant c_n$, then, by Lemma 16, we get

$$\frac{\|T^{b_n+v_n} e_j\|_{N_n}}{\|e_j\|_{N_n+1}} \leqslant \left(2Q^4\right)^{b_n+v_n} \frac{A_{N_n, j}}{A_{N_n+1, j}} \overset{(22)}{\leqslant} 1. \qquad \square$$

Theorem 21. *Every non-zero vector $x \in \lambda^1(A)$ is a cyclic vector with respect to the operator $T : \lambda^1(A) \to \lambda^1(A)$ defined in Corollary* 17.

Proof. It suffices to show that any vector x with $\|x\|_1 = 1$ is cyclic.

Because the finite sequences are dense in $\lambda^1(A)$, by Proposition 11(iii), e_0 is cyclic for T. Hence, by Proposition 8, it suffices to show that e_0 is an element of the closed linear span of the orbit of x.

Take any $N \in \mathbb{N}_+$. We will show that there is a polynomial P with $\|P(T)x - e_0\|_N \leqslant 6$.

Lemma 19 implies that we can find a number n, such that $N_n = N$, $y = \pi_{a_n+\Delta_n-1} x \in K_n$ and $\|\pi_{[a_n+\Delta_n,\infty)} x\|_{N+1} \leqslant 1$. By Corollary 15, we get a polynomial P with $|P| \leqslant C_n$ and sc $P \subseteq [v_n, a_n + \Delta_n)$ such that

$$\left\| \frac{T^{b_n}}{Q^{b_n}} P(T) y - e_0 \right\|_N \leqslant 4.$$

We have for the same polynomial P, that

$$\begin{aligned}\left\| \frac{T^{b_n} P(T)}{Q^{b_n}} x - e_0 \right\|_N &\leqslant \left\| \frac{T^{b_n} P(T)}{Q^{b_n}} y - e_0 \right\|_N + \left\| \frac{T^{b_n} P(T)}{Q^{b_n}} \pi_{[a_n+\Delta_n,\infty)} x \right\|_N \\ &\leqslant 4 + \left\| \frac{T^{b_n} P(T)}{Q^{b_n}} \pi_{[a_n+\Delta_n,\infty)} x \right\|_N.\end{aligned}$$

Let $P(t) = \sum_{l=v_n}^{a_n+\Delta_n-1} p_l t^l$. By Lemma 16, for any $z \in \lambda^1(A)$ we have that

$$\left\|T^{b_n} P(T) z\right\|_N = \left\| \sum_{l=0}^{a_n+\Delta_n-v_n-1} p_{l+v_n} T^l T^{b_n+v_n} z \right\|_N \leqslant C_n \left(2Q^4\right)^{a_n+\Delta_n} \left\|T^{b_n+v_n} z\right\|_N.$$

By Proposition 20, it now follows that

$$\begin{aligned}\left\| \frac{T^{b_n} P(T)}{Q^{b_n}} \pi_{[a_n+\Delta_n,\infty)} x \right\|_N &\leqslant \frac{C_n (2Q^4)^{a_n+\Delta_n}}{Q^{b_n}} \|T^{b_n+v_n} \pi_{[a_n+\Delta_n,\infty)} x\|_N \\ &\overset{(35)}{\leqslant} \|\pi_{[a_n+\Delta_n,\infty)} x\|_{N+1} \leqslant 2,\end{aligned}$$

so that

$$\left\| \frac{T^{b_n} P(T)}{Q^{b_n}} x - e_0 \right\|_N \leqslant 6.$$

Since the sequence of the unit balls is a basis of neighbourhoods of zero in $\lambda^1(A)$ (see Lemma 9), for every $M \in \mathbb{N}_+$ and every $\varepsilon > 0$ there is an $N \in \mathbb{N}_+$ such that

$$\left\{z \in \lambda^1(A)\colon\ \|z\|_N \leqslant 6\right\} \subseteq \left\{z \in \lambda^1(A)\colon\ \|z\|_M \leqslant \varepsilon\right\}.$$

This completes the proof of cyclicity of x with respect to $T : \lambda^1(A) \to \lambda^1(A)$. □

11. Comments

Remark 22. If a Köthe matrix A satisfies the condition (2), then $\lambda^1(A)$ is a Schwartz space, hence reflexive (see [7, 27.9, 27.10]). If A satisfies the conditions of Theorem 1, the space $\lambda^1(A)$ need not be nuclear, as the matrix $[j^{3-1/N}]$ shows.

The result does not cover spaces like the space of test functions $\mathcal{D}(\mathbb{R}^n)$, the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ or the space of smooth functions $C^\infty(\mathbb{R}^n)$. In fact, $\mathcal{D}(\mathbb{R}^n) \cong \bigoplus_{n\in\mathbb{N}} s$, $\mathcal{D}'(\mathbb{R}^n) \cong (s')^{\mathbb{N}}$, $C^\infty(\mathbb{R}^n) \cong s^{\mathbb{N}}$ (see [16]).

Remark 23. The reason why the result does not cover the space $H(\mathbb{C})$ (and $H(\{0\})$) comes from the fact that we cannot represent $H(\mathbb{C})$ as a Köthe sequence space with a matrix suitable for our proof and there is at least some topological reason for that, explained below.

Let $\alpha = (\alpha_j)$ be a sequence of positive numbers. A Fréchet space X is called *α-nuclear* if for any $N \in \mathbb{N}_+$ and a number $t > 0$ there is a number $K > N$ such that $e^{t\alpha_j}\delta_j(U_K, U_N) \xrightarrow[j\to\infty]{} 0$, where U_N is the unit ball of the N-th seminorm and $\delta_j(V, U)$ stands for the j-th Kolmogorov diameter, i.e. the infimum of all $\delta > 0$ such that with some at most j-dimensional subspace $F \subset X$ we have $V \subseteq \delta U + F$. It can be shown that α-nuclearity is a topological invariant for power series spaces (see [8, 9.3.2]).

A Köthe matrix $A = [A_{N,j}]$ is called regular if for any N the sequence $(\frac{A_{N,j}}{A_{N+1,j}})_j$ is monotonic. Then in the space $\lambda^1(A)$ we have that $\delta_j(U_K, U_N) = \frac{A_{N,j}}{A_{K,j}}$ for $K \geqslant N$ (see [8, 9.1.3]).

Fact 24. *Let $A = [A_{N,j}]$ be a Köthe matrix such that $\lambda^1(A)$ is (j)-nuclear and*

(i) *for each $N \in \mathbb{N}_+$ the sequence $(A_{N,j})_j$ tends monotonically to $+\infty$;*
(ii) *A is regular.*

Then there is no number $M > 0$ such that for each $N \in \mathbb{N}_+$ $\limsup_{j\to\infty} \frac{A_{N,j+1}}{A_{N,j}} \leqslant M$.

Proof. Assume to the contrary that such a number M exists. Take $Q > M$. By regularity, we have that for any $K \geqslant N$

$$\frac{\delta_{j+1}(U_K, U_N)}{\delta_j(U_K, U_N)} = \frac{A_{N,j+1}}{A_{K,j+1}} \frac{A_{K,j}}{A_{N,j}} \geqslant \frac{A_{K,j}}{A_{K,j+1}} \geqslant \frac{1}{Q}$$

for j big enough. It follows that $\delta_j(U_K, U_N) \geqslant D_{K,N}/Q^j$ for some number $D_{K,N}$.

Take $N \in \mathbb{N}_+$, $t > \log Q$. Then for any $K \geqslant N$

$$e^{tj}\delta_j(U_K, U_N) \geqslant D_{K,N}\left(\frac{e^t}{Q}\right)^j \xrightarrow[j\to\infty]{} \infty,$$

which contradicts (j)-nuclearity. □

We have that $H(\mathbb{C}) \cong \lambda^1([N^j])$ and as the matrix $[N^j]$ is regular, it is easy to check that $H(\mathbb{C})$ is (j)-nuclear. From Fact 24, we cannot find a regular Köthe matrix A such that $\lambda^1(A) \cong H(\mathbb{C})$ and A fulfils the assumptions of Theorem 1.

Acknowledgments

The paper is a part of the author's Ph.D. in preparation and the author would like to express his deepest gratitude to Professor P. Domański for suggesting this research and many valuable suggestions. The author would also like to thank the anonymous referee for deep remarks.

References

[1] A. Atzmon, An operator without invariant subspaces on a nuclear Fréchet space, Ann. of Math. (2) 117 (3) (1983) 669–694.

[2] F. Bayart, É. Matheron, Dynamics of Linear Operators, Cambridge Tracts in Math., vol. 179, Cambridge University Press, Cambridge, 2009.

[3] S.W. Brown, Some invariant subspaces for subnormal operators, Integral Equations Operator Theory 1 (3) (1978) 310–333.

[4] P. Enflo, On the invariant subspace problem for Banach spaces, Acta Math. 158 (3–4) (1987) 213–313.

[5] V.I. Lomonosov, Invariant subspaces of the family of operators that commute with a completely continuous operator, Funkcional. Anal. i Priložen. 7 (3) (1973) 55–56.

[6] D.H. Luecking, L.A. Rubel, Complex Analysis, a Functional Analysis Approach, Universitext, Springer-Verlag, New York, 1984.

[7] R. Meise, D. Vogt, Introduction to Functional Analysis, Oxf. Grad. Texts Math., vol. 2, The Clarendon Press, Oxford University Press, New York, 1997.

[8] A. Pietsch, Nuclear Locally Convex Spaces, Ergeb. Math. Grenzgeb., vol. 66, Springer-Verlag, New York, 1972, translated from the second German edition by William H. Ruckle.

[9] C.J. Read, A solution to the invariant subspace problem, Bull. Lond. Math. Soc. 16 (4) (1984) 337–401.

[10] C.J. Read, A short proof concerning the invariant subspace problem, J. Lond. Math. Soc. (2) 34 (2) (1986) 335–348.

[11] C.J. Read, The invariant subspace problem for a class of Banach spaces, II. Hypercyclic operators, Israel J. Math. 63 (1) (1988) 1–40.

[12] S. Rolewicz, Metric Linear Spaces, 2nd edition, PWN—Polish Scientific Publishers, Warsaw, 1984.

[13] W. Śliwa, The invariant subspace problem for non-Archimedean Banach spaces, Canad. Math. Bull. 51 (4) (2008) 604–617.

[14] W. Śliwa, An explicit example concerning the invariant subspace problem for Banach spaces, Rocky Mountain J. Math. 40 (2) (2010) 627–641.

[15] H.-G. Tillmann, Randverteilungen analytischer Funktionen und Distributionen, Math. Z. 59 (1953) 61–83.

[16] D. Vogt, Sequence space representations of spaces of test functions and distributions, in: Functional Analysis, Holomorphy, and Approximation Theory, Rio de Janeiro, 1979, in: Lect. Notes Pure Appl. Math., vol. 83, Dekker, New York, 1983, pp. 405–443.

[17] V.P. Zaharjuta, Spaces of functions of a single variable that are analytic in open sets and on compacta, Mat. Sb. (N.S.) 82 (124) (1970) 84–98.

[18] V.P. Zaharjuta, Extremal plurisubharmonic functions, Hilbert scales, and the isomorphism of spaces of analytic functions of several variables, I, II, Teor. Funkciĭ Funkcional. Anal. i Priložen. 19 (Vyp. 19) (1974) 133–157, 161; Teor. Funkciĭ Funkcional. Anal. i Priložen. 21 (1974) 65–83, 127.